Solve Initial Value ODE Using Explicit Euler Method

Objectives

- Solve a First order ODE Using Explicit Euler Method
- Explicit Euler Method is a Finite Difference Method (FDM).
- To provide some introduction to FDM.
- We will introduce Taylor series expansion.

Finite Difference Method (FDM)

- Many complex ordinary differential equations cannot be easily solved, analytically and we need to resort to numerical methods.
- FDM is a numerical method that converts the ordinary differential equations to algebraic equations.
- We can then solve the algebraic equations to obtain solutions.
- Finite difference methods like other numerical methods produce approximate solutions against exact solutions obtained using analytical methods.
- For solving many real world problems we don't need exact solutions in general but accurate solutions are preferred.

Steps in Finite Difference Method (FDM)

- The objective of a finite difference method for solving an ordinary differential equation (ODE) is to transform a calculus problem into an algebra problem by:
- Discretizing the continuous physical domain into a discrete finite difference grid
- Approximating the exact derivatives in the initial-value ODE by algebraic finite difference approximations (FDAs)
- Substituting the FDAs into the ODE to obtain an algebraic finite difference equation (FDE)
- Solving the resulting algebraic FDE

Taylor Series Expansion

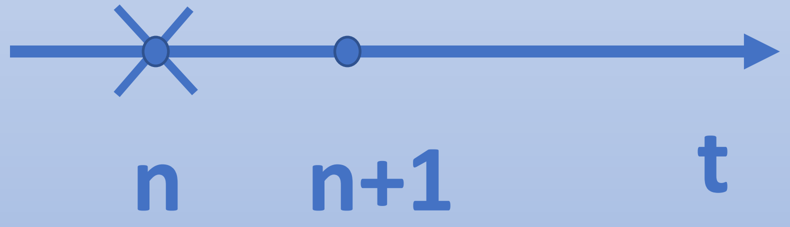
- Before we use Finite Difference Method, we need to understand Taylor series expansion of continuous functions.
- See below some very brief info on Taylor series which is an infinite series expansion of any function.
- $f(x+\Delta x) = f(x) + f'(x) * \Delta x + f''(x) * \frac{\Delta x^2}{2!} + \ldots\ldots \quad (1)$
- $f(x-\Delta x) = f(x) - f'(x) * \Delta x + f''(x) * \frac{\Delta x^2}{2!} - \ldots\ldots.. \quad (2)$
- Eq (1) is called forward Taylor series expansion and Eq (2) is called backward Taylor series expansion.
- Note Δx is a fraction and has a small value. Δx^2, Δx^3.. etc are even smaller and hence the terms associated with them can be ignored in general.

Explicit Euler Method

- $y_{n+1} = y_n + y'|_n * \Delta t + \frac{y''|_n}{2!} * \Delta t^2 + \frac{y'''|_n}{3!} * \Delta t^3$ ....(3)
- $y'|_n = \frac{y_{n+1} - y_n}{\Delta t} - \left(\frac{1}{2}\right) * y''|_n * \Delta t - \left(\frac{1}{6}\right) * y'''|_n * \Delta t^2 -$ ....(4)
- If we truncate the remainder terms, we have,
- $y'|_n = \frac{y_{n+1} - y_n}{\Delta t} - \left(\frac{1}{2}\right) * y''(\tau) * \Delta t$ ....(5) where $t < \tau < t + \Delta t$
- $y'|_n = \frac{y_{n+1} - y_n}{\Delta t}$ $O(\Delta t)$ ....(6)
- Equation (6) is a first-order forward-difference approximation of y’ at grid point n.

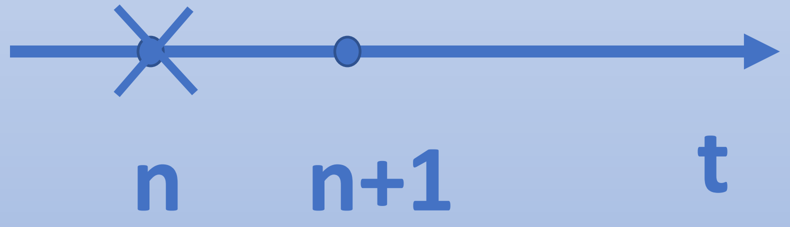
Explicit Euler Method

- Consider a general nonlinear first order ODE of the form
- $y' = f(t, y)$, $y(t_0) = y_0$ ....(7)
- $y'|_n = \frac{y_{n+1} - y_n}{\Delta t}$ - $\left(\frac{1}{2}\right) * y''(\tau_n) * \Delta t$ ....(5)
- Substitute Eq(5) into Eq(7)
- $\frac{y_{n+1} - y_n}{\Delta t}$ - $\left(\frac{1}{2}\right) * y''(\tau_n) * \Delta t = f(t_n, y_n)$ .....(8)
- Note n is the base point



Explicit Euler Method

- Solving Eq (8), we get
- $y_{n+1} = y_n + \Delta t * f(t_n, y_n) + \left(\frac{1}{2}\right) * y''(\tau_n) * \Delta t^2$;
- Let $f_n = f(t_n, y_n)$
- $y_{n+1} = y_n + \Delta t * f_n$ $\quad O(\Delta t^2)$ ……….(9)
- By repeated application after N steps
- $y_{n+1} = y_n + \Delta t * f_n$ $\quad O(\Delta t)$ ……….(10)
- Eq (10) is the (Finite Difference Equation) FDE of the Euler's method



Explicit Euler Method

- The FDE is explicit, since f_n does not depend on y_{n+1}
- The FDE requires only one known point. Hence, it is a single point method.
- The FDE requires only one derivative function evaluation [i.e.,f (t, y)] per step.
- The error in calculating y_{n+1} for a single step, the local truncation error, is $0(\Delta t^2)$.
- The global (i.e., total) error accumulated after N steps is $0(\Delta t)$.
- The explicit Euler method is conditionally stable.
- For a linear first-order homogeneous ODE of the form $y' + \alpha * y = 0$, the stability criteria is $\Delta t <= \frac{2}{\alpha}$

Definitions in Finite Difference Approximations

- Consistency – A FDE is *consistent* with an ODE if the difference between them (i.e., the truncation error) vanishes as Δt → 0. In other words, the FDE approaches the ODE.
- Order - The *order* of a FDE is the rate at which the global error decreases as the grid size approaches zero.
- Stability - A FDE is *stable* if it produces a bounded solution for a stable ODE and is *unstable* if it produces an unbounded solution for a stable ODE.
- Convergence - A finite difference method is *convergent* if the numerical solution of the FDE (i.e., the numerical values) approaches the exact solution of the ODE as Δt → 0.
- If a FDE demonstrates consistency and demonstrates conditional stability, we can say we can say that the method is convergent.

Explicit Euler Method

- Example ODE Problem
- $\frac{dy}{dx}$ = - 2 * x^3 + 12 * x^2 - 20 * x + 8.5
- From x = 0 to 4 with a step size of 0.5;
- The initial condition at *x* = 0 is *y* = 1.
- The exact solution is given as
- y = - 0.5 * x^4 + 4 * x^3 - 10 *x^2 + 8.5 * x + 1

Explicit Euler Method

- $y_{n+1} = y_n + \Delta t * f_n$ (Note Independent variable can be x or t)
- $f_n = - 2 * x^3 + 12 * x^2 - 20 * x + 8.5$; $[f_n = f(x_n, y_n)]$
- Let n = 0;
- $y_1 = y_0 + \Delta t * f_0$
- $\Delta t = \Delta x = 0.5$; $y_0 = 1$;
- $f_0 = f(x_0, y_0) = - 2 * 0^3 + 12 * 0^2 - 20 * 0 + 8.5 = 8.5$
- $y_1 = 1 + 0.5 * 8.5 = 5.25$
- Likewise, y_2, y_3 etc can be evaluated

Summary

In this video,

- We presented Explicit Euler Method to solve an Initial Value ODE
- The FDE is explicit, since f_n does not depend on y_{n+1}
- The explicit Euler method is conditionally stable.
- The global error is 0(Δt).
- The error can be minimized by using smaller steps.
- In the next video we can look at Implicit Euler Method to solve Initial Value ODE